Midterm Exam Calculus 2

17 March 2017, 9:00-11:00

university of groningen

The midterm exam consists of 4 problems. You have 120 minutes to answer the questions. You can achieve 100 points which includes a bonus of 10 points.

1. [**10+10+5 Points**] Let the function $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as

$$f(x,y) = \begin{cases} \frac{x^2y^2}{x^4+y^4} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases} .$$

   (a) Is f continuous at $(x,y) = (0,0)$? Justify your answer.

   (b) Use the definition of directional derivatives to determine for which unit vectors $\boldsymbol{u} = (v,w) \in \mathbb{R}^2$ the directional derivative $D_{\boldsymbol{u}}f(0,0)$ exists.

   (c) Is f differentiable at $(x,y) = (0,0)$? Justify your answer.

2. [**10+10 Points**] Consider the curve parametrized by $\mathbf{r} : [0, 2\pi] \to \mathbb{R}^3$ with

$$\mathbf{r}(t) = (\sin t - t\cos t)\,\mathbf{i} + 2\,\mathbf{j} + (\cos t + t\sin t)\,\mathbf{k}.$$

   (a) Determine the parametrization by arc length.

   (b) At each point on the curve, determine the curvature of the curve.

3. [**5+10+10 Points**] Consider the ellipsoid

$$x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 3$$

   which contains the point $(x_0, y_0, z_0) = (1, 2, 3)$.

   (a) Determine the tangent plane of the ellipsoid at the point (x_0, y_0, z_0).

   (b) Show that near the point (x_0, y_0, z_0) the ellipsoid is locally given as the graph of a function over the (x,y) plane, i.e. there is a function $f : (x,y) \mapsto z$ such that near (x_0, y_0, z_0) the ellipsoid is locally given by $z = f(x,y)$. Compute the partial derivatives $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ and show that the graph of the linearization of f at (x_0, y_0) agrees with the tangent plane found in part (a).

   (c) Use the method of Lagrange multipliers to find the points closest to and farthest away from the origin.

4. [**20 Points**] Determine

$$\iiint_W (2 + \sqrt{x^2+y^2})\,\mathrm{d}V,$$

   where

$$W = \{(x,y,z) \in \mathbb{R}^3 | \sqrt{x^2+y^2} \leq \frac{z}{2} \leq 3\}.$$